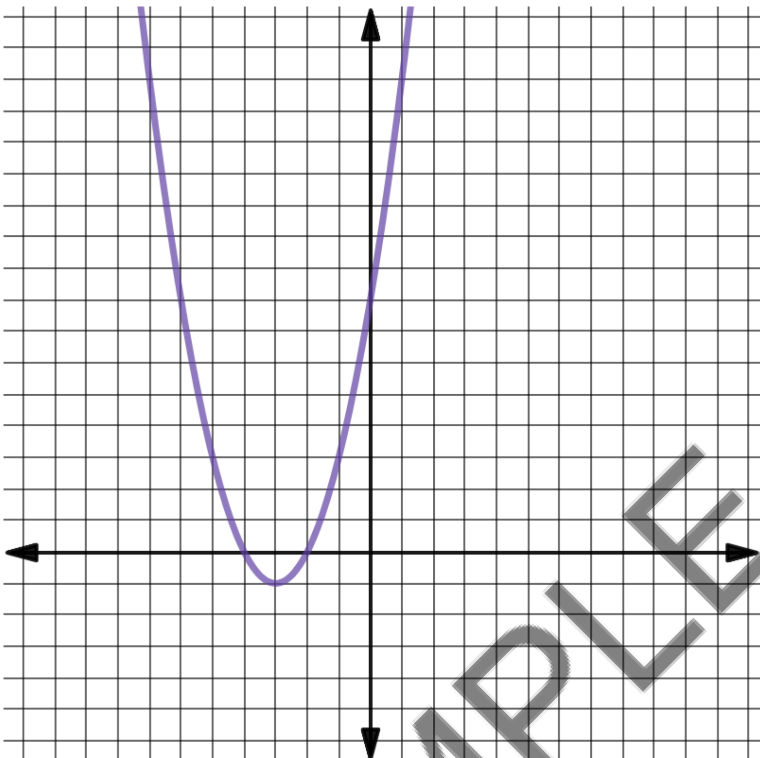
Chapter 10:
The Algebra of Quadratics

Lesson 1: Factoring to Find the Zeros

In Chapter 9, you saw that the x-intercepts were the zeros of a parabola. However, you will not always have a graph readily available, so you will have to find the zeros algebraically. The first method you will be working with in order to find the zeros is the factoring method. This method is most useful when the zeros are integers.

Exercise #1: The graph of the parabola below represents the quadratic equation $y = x^2 + 6x + 8$.



a) Factor the quadratic using your knowledge of factoring from Chapter 7.

b) Do you see a connection between the factors and the zeros? What is the connection?

c) Consider the statements below:

- If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.
- Similarly, if $(x+4)(x+2) = 0$, then either $x + 4 = 0$ or $x + 2 = 0$.
- So, you must solve for x by setting each factor equal to 0.

Exercise #2: Factor to find the zeros of the following quadratics. (Beginner)

a) $y = x^2 + 4x - 12$

b) $g(x) = x^2 - 2x - 120$

c) $f(x) = x^2 + 4x - 32$

Exercise #3: Factor to find the zero(s) of the following quadratics. (Intermediate)

a) $y = x^3 - 81x$

b) $h(x) = x^2 + 6x + 9$

c) $k(x) = 4x^2 - 25$

Exercise #4: Factor to the find the zeros of the following quadratics. (Advanced)

a) $j(x) = 2x^2 - 2x - 84$

b) $p(x) = 3x^2 - 4x - 15$

c) $q(x) = 5x^2 - 6x - 8$

Exercise #5: If the zeros of a quadratic are $x = \{2, -7\}$, what equation could represent this function?

Lesson 1 Extra Practice

EP1. Solve the following quadratics by factoring.

a) $y = x^2 + 3x + 2$

b) $y = 4x^2 + 9x + 5$

c) $y = 7x^2 + 2x$

d) $y = 3x^2 + 27x - 30$

EP2. Write a quadratic equation with integer coefficients that has the given zeros.

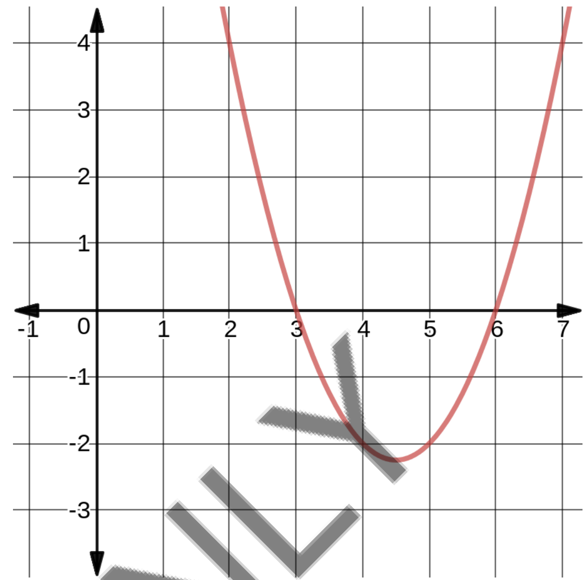
a) $x = \{-10, 5\}$

b) $x = \{0, 4\}$

c) $x = \frac{-3}{2}$ and $x = 2$

d) $x = \frac{1}{3}$ and $x = \frac{-5}{2}$

EP3. What is one possible equation of the parabola shown?



EP4. Solve each quadratic by factoring. Make sure the equation is equal to zero before you attempt to factor.

a) $x^2 + 8x = -15$

b) $-4x^2 - 8x - 3 = -3 - 5x^2$

EP5. Think critically and find all of the zeros of the following polynomial.

$$y = x^6 - 64$$

Lesson 2: Vertex Form & Completing the Square

In Algebra, you will use an algebraic method known as completing the square. By completing the square, you can find the zeros of the parabola, and also place the equation in **vertex form**. Today, you will practice placing equations in vertex form in order to find the vertex of a parabola.

A quadratic equation is in vertex form when it is in the form:

$$y = a(x - h)^2 + k$$

where (h, k) is the vertex of the parabola.

Exercise #1: Use the Method of Completing the Square to place the following quadratic equation in vertex form.

$$y = x^2 - 6x - 16$$

Step 1: Set the equation equal to 0.

$$x^2 - 6x - 16 = 0$$

Step 2: Using the addition property of equality, move the c term to the other side of the equation.

$$x^2 - 6x \quad = 16$$

Step 3: Divide the b term by 2, then square it (Example: $(\frac{-6}{2})^2 = 9$). Add that value to *both* sides of the equation, in order to preserve equality.

$$x^2 - 6x + 9 = 16 + 9$$

Step 4: Factor the left side of the equation, combine terms on the right side of the equation.

$$(x - 3)^2 = 25$$

Step 5: Move the constant to the other side and replace the 0 with y.

$$y = (x - 3)^2 - 25$$

Voila! Vertex Form of a Parabola!

Using your knowledge of quadratic transformations, why do you think this is called vertex form?

Exercise #2: Write each of the following quadratic equations in vertex form. Then, state the vertex.

a) $y = x^2 + 10x + 8$

b) $y = x^2 - 18x - 60$

c) $y = x^2 - 4x + 6$

d) $f(x) = x^2 + 12x + 20$

e) $g(x) = x^2 - 16x + 64$

f) $h(t) = t^2 - 3t + 4$

Lesson 2 Extra Practice

EP1. Rewrite the following quadratic equations in vertex form.

a) $y = x^2 + 10x + 22$

b) $f(x) = x^2 - 16x - 6$

c) $g(x) = x^2 + 5x - 12$

d) $h(t) = t^2 - 14t + 28$

EP2. State the vertex of each of the following quadratic equations.

a) $y = -3(x-4)^2 + 1$

b) $\frac{1}{2}y - 5 = (x+3)^2 - 2$

c) $y = x^2 - 6$

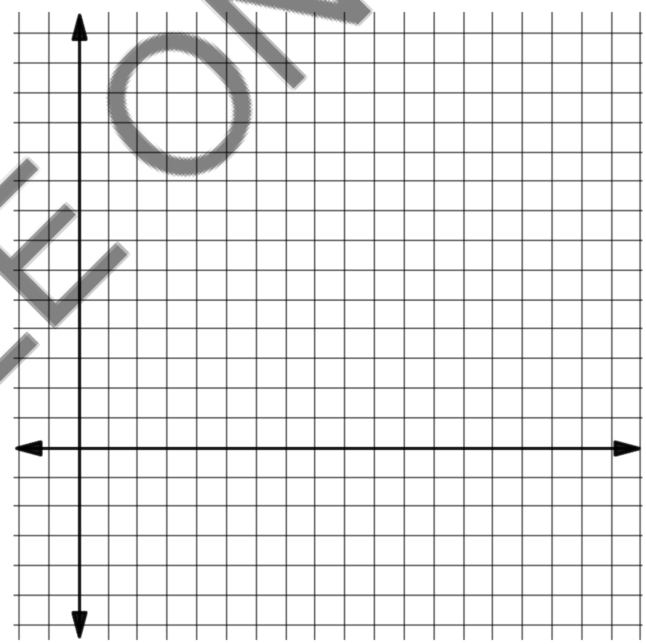
EP3. Consider the quadratic equation $p(x) = x^2 - 18x + 80$.

a) Rewrite $p(x)$ in vertex form.

b) What is the vertex of $p(x)$?

c) Find the equation of the axis of symmetry algebraically.

d) Graph $p(x)$ on the set of axes below.



EP4. The quadratic equation $f(x) = -2(x+5)^2 - 3$ is written in vertex form.

a) What is the vertex of $f(x)$?

b) Rewrite the equation in standard form.

Lesson 3: Solving and Simplifying Square Roots

Completing the square is not only a method used to find the vertex of a parabola, but it is also a useful method to find the zeros of a parabola as well. Before you can find the zeros by completing the square, it is important to understand how square roots could be used as inverse operations.

Exercise #1: Solve for both values of x by using a square root as an inverse operation.

a) $x^2 = 25$

b) $x^2 = 64$

c) $x^2 = 121$

Exercise #2: Solve each multi-step equation using a square root as an inverse operation.

a) $3x^2 + 10 = 58$

b) $\frac{1}{2}x^2 - 7 = 43$

c) $(x+3)^2 - 5 = 20$

In Exercises #1 and #2, the solutions were all integers. This will not always be the case and therefore, you will need to work with simplifying square roots. You will begin by exploring basic operations of square roots.

Exercise #3: Simplify the following expressions. The first one has been done for you.

a) $\sqrt{20} \cdot \sqrt{5}$

$\sqrt{20 \cdot 5}$

$\sqrt{100}$

± 10

b) $\sqrt{8} \cdot \sqrt{18}$

c) $\sqrt{2} \cdot \sqrt{98}$

Exercise #4: Square roots can be multiplied by square roots, as seen in Exercise #3. Likewise, square roots could be split into factors in order to write the square root in **simplest radical form**. Using part (a) as a guide, simplify the following square roots.

a) $\sqrt{72}$

$\sqrt{36 \cdot 2}$

$\sqrt{36} \cdot \sqrt{2}$

$\pm 6\sqrt{2}$

b) $\sqrt{288}$

c) $\sqrt{192}$

d) $\sqrt{240}$

e) $\sqrt{90}$

f) $\sqrt{84}$

Exercise #5: Now that you have practiced with some simple examples, try some more challenging examples below.

a) $4\sqrt{18}$

b) $8\sqrt{216}$

c) $4\sqrt{7} + \sqrt{448}$

Lesson 3 Extra Practice

EP1. Solve each of the following using square roots as inverse operations.

a) $4x^2 - 21 = -5$

b) $-5x^2 + 13 = -32$

c) $(x-4)^2 + 6 = 15$

d) $(x+3)^2 - 10 = 134$

EP2. Express each square root in simplest radical form.

a) $\sqrt{245}$

b) $\sqrt{585}$

EP3. Express each of the following in simplest radical form.

a) $3\sqrt{3} + 10\sqrt{3}$

b) $90\sqrt{2} + \sqrt{50}$

EP4. Solve each of the following using square roots as inverse operations. Express your answer in simplest radical form.

a) $12x^2 - 15 = 741$

b) $2x^2 + 10 = 310$

c) $3x^2 + 6 = 1095$

d) $(x - 1)^2 - 6 = 354$

Lesson 4: Finding Zeros by Completing the Square

In Lesson 2 you practiced completing the square and in Lesson 3 you placed square roots in simplest radical form. Today, you will combine the ideas from both of those lessons and find the zeros of a quadratic by completing the square.

Exercise #1: Begin by finding the solutions to the equation below, using square roots as inverse operations. Express your answer in simplest radical form.

$$(x+2)^2 - 8 = 0$$

Notice that this equation is already in vertex form. Now, you will have to place the equation in vertex form, and then find the zeros of the quadratic.

Exercise #2: For the following equations, use the method of completing the square to find the zeros.

a) $x^2 - 2x - 15 = 0$

b) $x^2 + 2x = 35$

Exercise #3: Now, find the zeros of each quadratic using the method of factoring. How do your solutions compare to the solutions in Exercise #2?

a) $x^2 - 2x - 15 = 0$

b) $x^2 + 2x = 35$

Exercise #4: The solutions in Exercise #2 and Exercise #3 are the same, so why is it essential to learn this new method of completing the square to find the zeros? This is because not all quadratics are able to be factored. For quadratics that cannot be factored, you will use the method of completing the square. Find the zeros of the quadratic equations below.

a) $x^2 + 4x - 3 = 0$

b) $x^2 + 6x - 59 = 0$

Lesson 4 Extra Practice

EP1. Find the zeros for each of the quadratic equations below.

a) $(x+6)^2 - 63 = 0$

b) $y = (x-1)^2 - 40$

EP2. Use the method of completing the square to find the zeros of each quadratic equation below. Express your answers in simplest radical form.

a) $x^2 - 4x - 91 = 0$

b) $x^2 + 14x = -37$

EP3. Find the zeros and the vertex of each of the following quadratics using the method of completing the square.

a) $x^2 - 8x + 15 = 0$

b) $x^2 + 12x - 54 = 0$

c) $x^2 + 15x + 15 = x + 2$

d) $x^2 - 8x + 8 = 0$

Lesson 5: The Quadratic Formula

The method of completing the square is most convenient when the a term is equal to 1 and the b term is an even number. As you learned in previous chapters, the standard form of a quadratic is $ax^2 + bx + c$. When the method of completing the square is performed on the standard form of a quadratic equation, the formula below is derived. This is known as the **quadratic formula**. The quadratic formula is most useful when the a term does not equal 1 and the quadratic cannot be factored.

Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In order to find the zeros of any quadratic in standard form, substitute the a, b, and c values into the formula above.

Exercise #1: Determine the zeros of the following quadratics using the quadratic formula.

a) $2x^2 - 3x - 5 = 0$

b) $4x^2 + 8x + 5 = 2$

c) $3x^2 + 9x = x^2 - 7$

d) $5x^2 + 9x + 4 = 0$

Exercise #2: For each of the following quadratics, determine the exact zeros. Express your solution in simplest radical form.

a) $9x^2 + 4x - 16 = 0$

b) $9x^2 - 6x - 11 = 0$

c) $14x^2 + 1 = 6x^2 + 7x$

d) $4x^2 + 4x - 8 = 1$

Exercise #3: Consider the quadratic equation $y = x^2 + 5x + 5$.

a) Find the zeros using the quadratic formula. Express your answer in simplest radical form.

b) Find the vertex of the equation by completing the square and placing in vertex form.

Lesson 5 Extra Practice

EP1. Find the solutions, if such exist, to the following quadratic equations using the quadratic formula. Express your answer in simplest radical form.

a) $3x^2 - 5x - 8 = 0$

b) $x^2 - 9 = 0$

c) $10x^2 - 9x + 6 = 0$

d) $6x^2 - 11 = 0$

e) $3x^2 + 9x - 15 = 0$

f) $8x \cdot (2x - 1) = -1$

EP2. Using the following quadratic equation, answer each question using all previous knowledge from this chapter.

$$y = x^2 + 7x + 12$$

a) Find the axis of symmetry, algebraically.

b) Use the axis of symmetry to find the vertex.

c) Rewrite in vertex form, then state the vertex.

d) Using the equation in vertex form, find the zeros.

e) Solve by factoring.

f) Solve using the quadratic formula.

Lesson 6: Quadratic Word Problems

Your hard work with quadratics has led you to this moment. Chapters 7, 8, 9, and 10 have prepared you for the applications of quadratics that you will be working with today.

Begin each exercise in this lesson by attempting to factor first. If it cannot be factored determine if the most appropriate method for solving is completing the square, or the quadratic formula.

Exercise #1: The difference of the square of a number and 18 is equal to 3 times that number. What is the negative solution?

Exercise #2: The product of two consecutive odd integers is 195. Both integers are negative. What are the two integers?

Exercise #3: The product of two consecutive positive integers is 506. What is the sum of the integers?

Exercise #4: The width of a rectangle is 1 centimeter less than the length. The area of the rectangle is 30 square centimeters. What is the length, in centimeters, of the rectangle?

Exercise #5: The length of a rectangular picture frame is twice the width. The area is 50 square inches. Find the dimensions of the picture frame.

Exercise #6: The length of a rectangular pool is 4 meters less than twice the width. A walkway, 1 meter wide, is to be installed around the entire pool. Find the dimensions of the pool if the combined area of the pool and the walkway is 56 square meters.

Lesson 6 Extra Practice

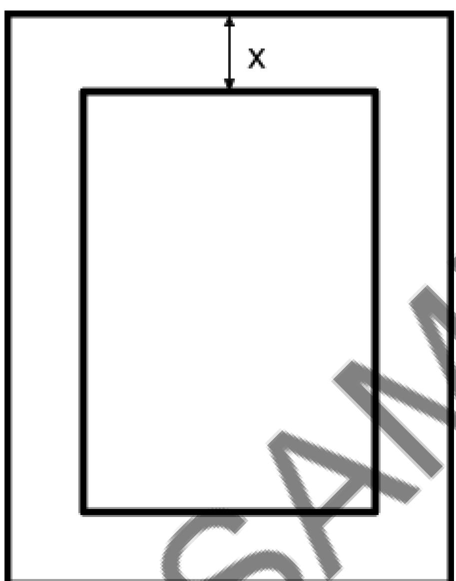
EP1. The product of two consecutive odd integers is 99. Find the integers.

EP2. The product of two consecutive integers is three less than three times their sum. Find the integers.

EP3. The width of a rectangle is 11 inches less than its length. Find the dimensions of the rectangle if the area is 80 square inches.

EP4. The length of a rectangular painting is 2 feet less than three times the width. Find the dimensions of the painting if the area is 65 square feet.

EP5. A rectangular picture has a frame around it. The frame adds x inches to the picture on all sides. The framed picture has a length of 30 inches and a width of 20 inches. If the area of the picture only is 336 square inches, what is the width of frame, x, in inches?



Lesson 7: More Quadratic Word Problems

Perhaps one of the most useful applications of quadratics is projectile motion. Since gravity is constant on Earth, anything that goes up must come down…and in a parabolic path! Today, you will work with projectile motion, analyzing different situations and interpreting the solutions.

Exercise #1: A ball is shot from a cannon into the air with an upward velocity of 40 feet per second. The equation that gives the height, h, of the ball at any time, t, is shown below.

$$h(t) = -16t^2 + 40t + 1.5$$

a) What is the maximum height the ball reaches? Justify your answer.

b) How many seconds will it take the ball to hit the ground? Round your answer to the nearest hundredth of a second.

c) How long will it take the ball to first reach a height of 15 feet? Round your answer to the nearest hundredth of a second.

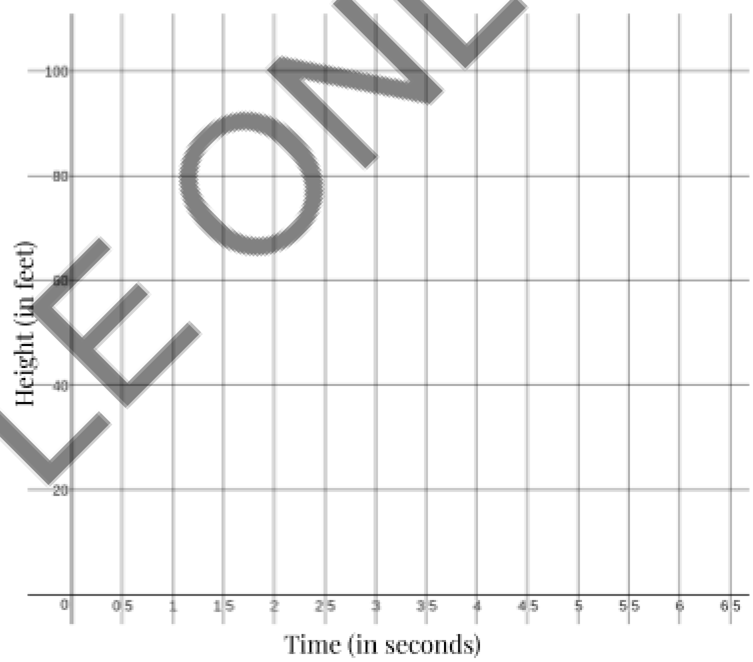
Exercise #2: A hiker drops his compass from the top of a 100-foot cliff. The height of compass as a function of time can be modeled by the equation $h(t) = -16t^2 + 100$, where $h(t)$ is the height of the compass above the ground after t seconds.

a) Interpret $h(1) = 84$.

b) Interpret the y-intercept.

c) How long will it take the compass to hit the ground? Justify your answer.

d) Sketch the path of the compass falling to the ground on the grid below.



Exercise #3: Rio is a diver on the diving team. The path of his dive could be modeled by the equation $y = -4.9x^2 + 3.14x + 1$, where y is Rio's height, in meters, x seconds after he leaves the diving board.

a) What is the maximum height Rio will reach? Round your answer to the nearest tenth of a meter.

b) How long will it take Rio to hit the water? Round your answer to the nearest hundredth of a second.

Lesson 7 Extra Practice

EP1. If a toy rocket is launched vertically upward from ground level with an initial velocity of 128 feet per second, then its height, h, after t seconds is given by the equation $h(t) = -16t^2 + 128t$.

a) How long will it take the rocket to reach the ground?

b) After how many seconds will the rocket first reach a height of 156 feet above the ground?

c) How long will it take the rocket to reach its maximum height?

d) What will the maximum height of the rocket be at that time?

EP2. A baseball is hit and travels the path given by the function $f(x) = -\frac{1}{125}x^2 + 3x + 3$, where x is the horizontal distance, in feet, the ball travels, and $f(x)$ represents the ball's vertical height above the ground.

a) What is the maximum height the baseball will reach?

b) Over what interval is the baseball's height increasing?

c) Explain the meaning of the y-intercept in the context of the problem.

d) What will be the total horizontal distance the baseball travels when it hits the ground. Round your answer to the nearest foot.

Chapter Review

<u>Part I Questions</u>: For each statement or question, choose the word or expression that, of those given, best completes the statement or answers the question.

CR1. Jared determines the zeros of the function $f(x)$ to be -3 and 2. What could be Jared's function?

1) $f(x) = (x-3)(x-2)$
2) $f(x) = (x-3)(x+2)$
3) $f(x) = (x+3)(x-2)$
4) $f(x) = (x+3)(x+2)$

CR2. What is the solution set of the equation $(x+6)(x+c) = 0$?

1) 6 and c
2) -6 and c
3) 6 and $-c$
4) -6 and $-c$

CR3. If $f(x) = 5x^2 - 19x - 4$, which equation can be used to determine the zeros of the function?

1) $0 = (5x+1)(x-4)$
2) $0 = (5x-1)(x+4)$
3) $0 = (5x+1)(5x-20)$
4) $0 = (5x-1)(5x+20)$

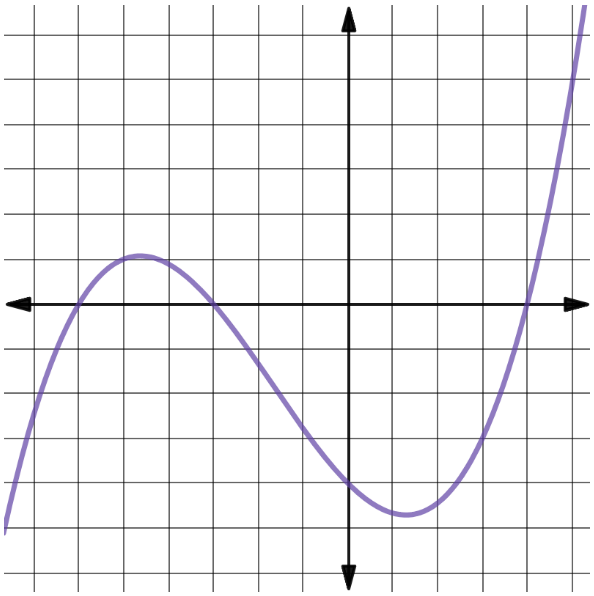
CR4. The zeros of the function $f(x) = 3x^2 + 18x - 21$ are

1) 7 and -1
2) 7 and 1
3) -7 and 1
4) -7 and -1

CR5. The zeros of the function $f(x) = (x+4)^2 - 36$ are

1) -10 and 2
2) -2 and 10
3) -6 and 4
4) -4 and 6

CR6. The graph of $y = f(x)$ is shown below.



Which set lists all the real solutions of $f(x) = 0$?

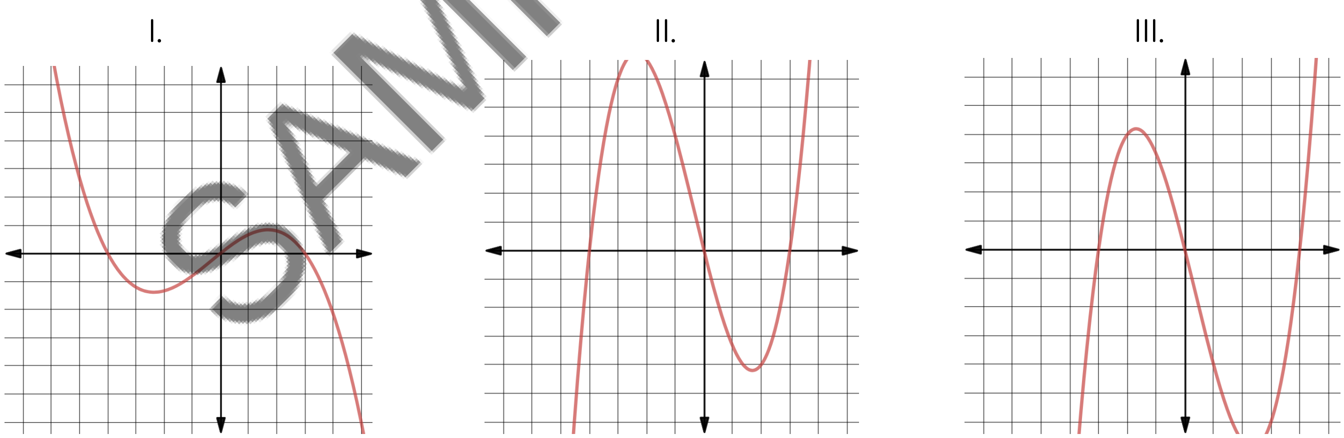
1) $\{4\}$

2) $\{-6, -4, -3, 4\}$

3) $\{-4, 3, 6\}$

4) $\{-6, -3, 4\}$

CR7. A polynomial function contains the factors x, $x+4$, and $x-3$. Which graph(s) below could represent the graph of this function?



1) I, only

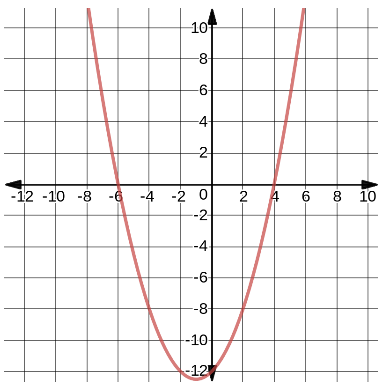
2) III, only

3) I and II

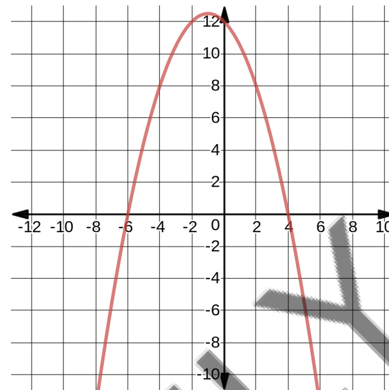
4) I, II, and III

CR8. The graphs below represent functions defined by polynomials. Which graph is represented by the function $g(x) = -\frac{1}{2}(x-4)(x+6)$?

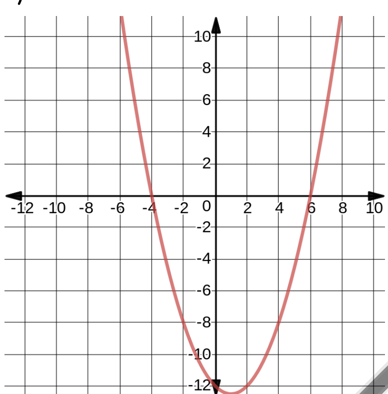
1)



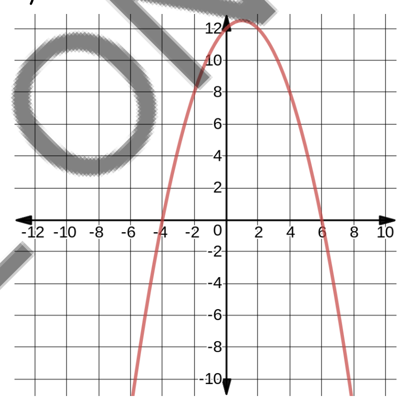
3)



2)



4)



CR9. Which value of x is a solution to the equation $25x^2 - 17 = 32$?

1) $\frac{49}{25}$

3) $-\frac{7}{5}$

2) $-\frac{49}{25}$

4) $\frac{5}{7}$

CR10. When solving the equation $x^2 - 6x - 11 = 0$ by completing the square, which equation is a step in the process?

1) $(x-3)^2 = 20$

3) $(x-6)^2 = 20$

2) $(x-3)^2 = 2$

4) $(x-6)^2 = 2$

CR11. The quadratic equation $x^2 - 8x = 17$ is rewritten in the form $(x + p)^2 = q$, where q is a constant. What is the value of p?

1) -4
2) 4
3) 33
4) 1

CR12. Which equation has the same solutions as $x^2 + 2x - 13 = 0$?

1) $(x+1)^2 = 12$
2) $(x-1)^2 = 12$
3) $(x-1)^2 = 14$
4) $(x+1)^2 = 14$

CR13. What are the solutions to the equation $x^2 - 12x = 24$?

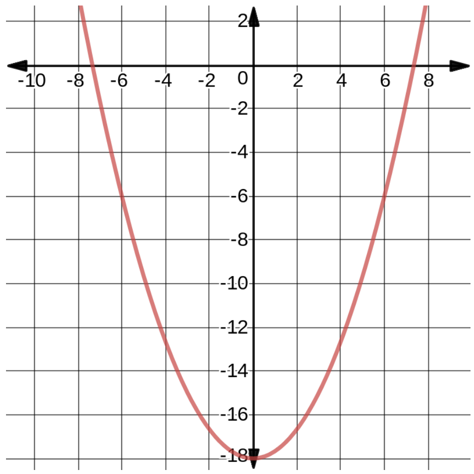
1) $x = 6 \pm 2\sqrt{15}$
2) $x = -6 \pm 2\sqrt{15}$
3) $x = 6 \pm 6\sqrt{10}$
4) $x = -6 \pm 6\sqrt{10}$

<u>Open Response Questions</u>: **For each question, clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc.**

CR14. Solve $4x^2 - 14x = 30$ by factoring.

CR15. Explain how to determine the zeros of $f(x) = (x)(x-4)(x+2)$. State the zeros of the function.

CR16. Massiel is given the graph of the function $y = \frac{1}{3}x^2 - 18$. She wants to find the zeros of the function, but is unable to read them exactly from the graph.



Find the zeros in simplest radical form.

CR17. The height, H, of an object dropped from the top of a building after t seconds is given by $H(t) = -16t^2 + 96$.

How many feet did the object fall between one and two seconds after it was dropped?

Determine, algebraically, how many seconds it will take for the object to reach the ground.

CR18. Solve the following equation by completing the square: $x^2 + 6x = 2x + 12$

CR19. Given the function $f(x) = 2x^2 + 11x - 7$, state whether the vertex represents a maximum or minimum point for the function. Explain your answer.

CR20. Solve for x to the *nearest tenth*: $2x^2 + 3x - 6 = 0$.

CR21. Solve the equation $6x^2 + 13x = 5$ algebraically for the exact value(s) of x.