Chapter 11: Sequences

Lesson 1: Arithmetic Sequences

In this lesson you will begin working with sequences. A **sequence** is an ordered list of numbers, where each number in the sequence is called a **term**.

Sequences often have patterns that allow you to predict any term in the sequence. The term you are trying to predict is called the n^{th} term, where n is any whole number.

Exercise #1: Based on the sequences below, determine a pattern within the sequence. After you have determined the pattern, find the 7^{th} term.

a) $6,\ 12,\ 18,\ 24,\ \ldots$

b) $2,\ 6,\ 10,\ 14,\ \ldots$

c) $-8,\ -1,\ 6,\ 13,\ \ldots$

d) $20,\ 18.5,\ 17,\ 15.5,\ \ldots$

Sequences that follow a pattern of repeated addition are referred to as **arithmetic sequences.** The pattern of how much is being added each time is called the **common difference.**

Exercise #2: Decide if each sequence below is an arithmetic sequence. If it is, state the common difference, then find the 8^{th} term.

a) $2,\ 4,\ 8,\ 16,\ \ldots$

b) $-12,\ -17,\ -22,\ -27,\ \ldots$

c) $-2,\ 1,\ 4,\ 7,\ \ldots$

d) $14,\ 34,\ 54,\ 64,\ \ldots$

In Exercises #1 and #2, you were asked to find the 7^{th} and 8^{th} term. Suppose you were asked to find the 50^{th} term. It may be tedious to add the same amount over and over, until you reach the 50^{th} term. The formula below is the arithmetic sequence formula where a_n is the n^{th} term, a_1 is the first term of the sequence, n is the term position, and d is the common difference.

Arithmetic Sequence Formula:

$$a_n = a_1 + (n-1) \cdot d$$

Exercise #3: For each of the arithmetic sequences below, develop a formula to find the n^{th} term. Then, find the 22^{nd} term.

a) $18,\ 8,\ -2,\ -12,\ \ldots$

b) $7,\ 8.75,\ 10.5,\ 12.25,\ \ldots$

Exercise #4: Given the following information, find the specified term.

a) $a_1 = 13$, $d = -2$
Find a_{31}

b) $a_1 = 11$, $d = 3$
Find a_{63}

c) $a_1 = 20$, $d = 0.25$
Find a_{14}

d) $a_3 = 10$, $d = 4$
Find a_{16}

Lesson 1 Extra Practice

EP1. For the following arithmetic sequences, determine a formula to find the n^{th} term of the sequence. Then, find the specified term.

a) $16,\ 13,\ 10,\ \ldots$
Find a_{12}

b) $3,\ 6.2,\ 9.4,\ \ldots$
Find a_{10}

EP2. Based on the information given about certain arithmetic sequences, answer the questions below.

a) Given: $a_{12} = 32$ and $a_{16} = 38$
Find d

b) Given: $a_4 = 15$ and $d = -4$
Find a_1

c) Given: $a_1 = -4$ and $d = 3$
Find a_n

d) Given: $a_4 = 15$ and $a_7 = 33$
Find a_1

EP3. Given the following arithmetic sequences, find the first five terms.

a) $a_n = 8 + (n - 1) \cdot 2$

b) $a_n = -2.2 + (n - 1) \cdot 0.1$

EP4. Mr. Lobenstein, the orchestra teacher is setting up for a concert. He arranges the chairs such that there are 5 chairs in the first row, 8 chairs in the second row, 11 chairs in the third row, and so on. How many chairs are in the 9th row?

EP5. Gavin has a running workout he will do for 20 days. He runs for 5 minutes on the first day, 7 minutes on the second day, 9 minutes on the third day, and so on. If he continues this schedule for 20 days, how many calories will he burn on day 20 if he burns 30 calories per minute of running?

Lesson 2: Geometric Sequences

In this lesson, you will work with **geometric sequences**. Geometric sequences are sequences where each term is found by multiplying the term before by a **constant factor**, or **common ratio**.

Exercise #1: Given the following sequences, determine if a common ratio exists. If one does exist, state the common ratio.

a) $3,\ 12,\ 48,\ 192,\ \ldots$

b) $-2,\ 0,\ 2,\ 4,\ \ldots$

c) $-3,\ 6,\ -12,\ 24,\ \ldots$

d) $625,\ 500,\ 400,\ 320,\ \ldots$

The main difference between arithmetic sequences and geometric sequences is the pattern in which they increase or decrease. Arithmetic sequences change by a common difference, while geometric sequences change by a common ratio.

Exercise #2: State whether each of the sequences below represent an arithmetic sequence, geometric sequence, or neither. Explain why.

a) $5,\ 10,\ 20,\ 40,\ \ldots$

b) $1,\ 5,\ 9,\ 12,\ \ldots$

c) $54,\ 18,\ 6,\ \ldots$

d) $-4,\ 1,\ 6,\ 11,\ \ldots$

The formula below is used to find the n^{th} term of a geometric sequence, where a_n represents the n^{th} term, a_1 represents the first term of the sequence, r represents the common ratio, and n is the term position.

Geometric Sequence Formula

$$a_n = a_1 \cdot r^{n-1}$$

Exercise #3: Using the geometric sequence formula, find the specified term.

a) $a_1 = 3$ and $r = 2$
Find a_8

b) $a_1 = 16$ and $r = \frac{1}{2}$
Find a_8

Exercise #4: For each sequence below, develop a formula to find the n^{th} term of the sequence. Then find the specified term.

a) $1024,\ 512,\ 256,\ \ldots$
Find a_{13}

b) $32805,\ 10935,\ 3645,\ \ldots$
Find a_7

c) $7,\ 42,\ 252,\ \ldots$
Find a_6

d) $8,\ 12,\ 18,\ \ldots$
Find a_7

Lesson 2 Extra Practice

EP1. Determine if each sequence is arithmetic, geometric, or neither. Then, write the formula that could be used to find the n^{th} term of the sequence.

a) $-3, \ -18, \ -108, \ -648, \ ...$

b) $2, \ 4, \ 12, \ 48, \ ...$

c) $-2, \ 6, \ -18, \ 54, \ ...$

d) $-35, \ 165, \ 365, \ 565, \ ...$

e) $1, \ 2, \ 6, \ 24, \ ...$

f) $1000, \ 100, \ 10, \ ...$

EP2. Based on the information given about certain geometric sequences, answer the questions below.

a) Given: $a_1 = 3.5$ and $r = -2$
Find: a_{10}

b) Given: $a_1 = 16$ and $a_3 = 4$
Find: a_7

EP3. Given the following geometric sequences, find the first five terms.

a) $a_n = 12 \cdot (2.5)^{n-1}$

b) $a_n = 9 \cdot (1)^{n-1}$

EP4. Two people know a secret. Every day the amount of people that know the secret triples. How many people will know the secret on the tenth day?

EP5. A scientist is working with a radioactive element with a half-life of one day (every day, half of the previous amount remains). If the element had an initial amount of 1000mg, how much of the element will remain after 8 days? Round your answer to the nearest tenth of a milligram.

EP6. Abby works at a job that starts out paying $15 per hour. Each year she continues at the job, she will get a 5% raise. How much will Abby make per hour if she stays at that job for 30 years?

Lesson 3: Recursive Sequences

The final type of sequence you will work with is called a **recursive sequence**. A recursive sequence is a sequence that requires computation of all previous terms in order to find a_n.

Exercise #1: One of the most well known sequences in mathematics is the Fibonacci sequence. The next number in this sequence is found by adding the two previous numbers together.

Fibonacci Sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

The recursive formula that could be used to represent this sequence is shown below:

$$a_n = a_{n-2} + a_{n-1}, \text{ where } a_1 = 1 \text{ and } a_2 = 1$$

For example, in order to find the sixth term of the sequence, you would substitute 6 in for n, as shown below.

$$a_6 = a_{6-2} + a_{6-1}$$

or

$$a_6 = a_4 + a_5$$

However, you do not know the fourth or fifth term of the sequence just yet. You would need to start with the third term since you were only given a_1 and a_2.

$$a_3 = a_2 + a_1$$

$$a_3 = 1 + 1$$

$$a_3 = 2$$

a) Find a_4

b) Find a_5

c) Find a_6

Exercise #2: Recursive sequences could take the form of arithmetic sequences, geometric sequences, or as you saw in Exercise #1, neither of those sequences. Use the following recursive formulas to find the n^{th} term of each sequence.

a) $a_n = a_{n-1} - 6$
If $a_1 = 2$, find a_5

b) $a_n = a_{n-1} \cdot 4$
If $a_1 = 0.5$, find a_6

c) $a_n = 2n \cdot a_{n-1}$
If $a_1 = 3$, find a_5

d) $a_n = 3 \cdot a_{n-1} + n$
If $a_1 = -1$, find a_7

Lesson 3 Extra Practice

EP1. Given the following recursive formulas, find the fifth term.

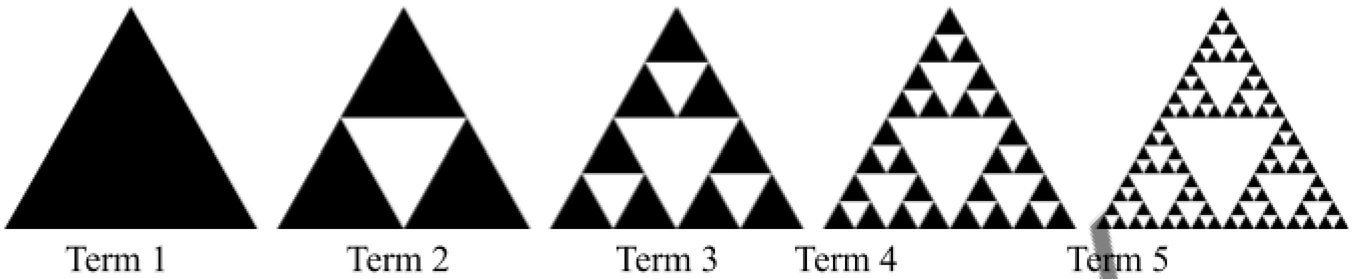
a) $a_1 = 4$
$a_{n+1} = (a_n)^2 - 10$

b) $a_1 = 5$
$a_n = a_{n-1} - n$

c) $a_1 = \frac{1}{2}$
$a_n = 2n + a_{n-1}$

d) $a_1 = 5$
$a_n = 2 \cdot a_{n-1} + n$

EP2. The sequence below represents Sierpinski's triangle. Sierpinski's triangle is a recursive image where the next image is found by dividing each equilateral triangle from the previous image into 3 smaller equilateral triangles.



The recursive formula used to represent Sierpinski's n^{th} triangle is:

$$f(1) = 1$$
$$f(n) = 3 \cdot f(n-1)$$

a) Use your knowledge of recursive sequences to prove the first five terms of Sierpinski's triangle match the formula given.

b) Develop a geometric sequence formula to find the number of triangles in the n^{th} term without using recursion.

Chapter Review

<u>Part I Questions</u>: For each statement or question, choose the word or expression that, of those given, best completes the statement or answers the question.

CR1. In a sequence, the first term is 7 and the common difference is 3. The sixth term of this sequence is

1) -11
2) -8
3) 22
4) 25

CR2. At a baseball stadium, the number of seats per row around home plate increases at a constant rate. If there are 40 seats in row 2 and 52 seats in row 6. How many seats are there in row 18?

1) 88
2) 91
3) 94
4) 97

CR3. If $a_n = (a_{n-1}) + 2n$ and $a_1 = 3$, what is the value of a_5?

1) 13
2) 21
3) 31
4) 43

CR4. For the sequence $-16,\ -5,\ 6,\ 17,\ \ldots,$ the expression that defines the n^{th} term where $a_1 = -16$ is

1) $-16 + 11n$
2) $-16 + 11(n-1)$
3) $11 - 16n$
4) $11 - 16(n-1)$

CR5. The fourth term in an arithmetic sequence is -4 and the sixth term is -22. If the first term is a_1, which is an equation for the n^{th} term of the sequence?

1) $a_n = 9n + 23$
2) $a_n = 9n + 32$
3) $a_n = -9n + 23$
4) $a_n = -9n + 32$

CR6. At a certain parking garage, the cost to park was $4 for up to one hour. Every additional half hour costs $1.50. Which recursive function could be used to determine the cost of parking in this garage for 6 hours?

1) $a_1 = 4;\ a_n = a_{n-1} + 1.50$

2) $a_1 = 0;\ a_n = 4a_{n-1} + 1.50$

3) $a_1 = 1.50;\ a_n = a_{n-1} + 4$

4) $a_1 = 4;\ a_n = 1.50a_{n-1} + 4$

CR7. If $f(1) = -2$ and $f(n) = 3f(n-1) - 1$, then $f(5) =$

1) -66

2) -201

3) -67

4) -202

__Open Response Questions__: **For each question, clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc.**

CR8. Determine and state whether the sequence 1000, 600, 360, 216, ... displays exponential behavior. Explain how you arrived at your decision.

CR9. Write the first four terms of the recursive sequence defined below.

$$a_1 = 3$$

$$a_n = (a_{n-1})^2 - (a_{n-1}),\ \text{for } n > 1$$